K.L.E Society's
S. Nijalingappa College
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PG Department of Mathematics
QUESTION BANK

Complex Analysis

1. Define harmonic conjugate. Show that the two functions $u(x,y)$ and $v(x,y)$ are harmonic conjugates to each other if and only if they are constants.

2. Derive Cauchy's integral formula and hence evaluate

$$\int_C \frac{z^2}{(z-1)(z-2)} dz \text{ where } C: |z| = 3.$$

3. State and prove fundamental theorem of algebra.
4. Expand the following function in Laurent's series valid for the region $2 < |z| < 3$.

$$f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$$

5. Find the radius of convergence of the following functions.

(i) $\sum \left(1 + \frac{1}{n}\right)^{n^2} z^n$

(ii) $\sum \left(\frac{1}{2^n + 1}\right) z^n$

6. Expand $\log(1+z)$ using Taylor's series about $z = 0$ and hence deduce the expansion of $log\sqrt{\frac{1+z}{1-z}}$
7. State and prove Laurent's theorem.

Define the following terms and give one example for each.

(i) Pole

(ii) Removable singularity

(iii) Essential singularity

(iv) Isolated singularity 4

(b) Show that a function which has no singularity in the finite part of the complex plane and has a pole of order n at infinity is a polynomial of degree n. 6

(c)Show that the function $f(z) = \frac{z^2+4}{e^z}$ has isolated essential singularity at $z = \infty$. **4**

5. (a) State and prove Cauchy's residue theorem. **6**

(b)Find the residue of the function

$f(z) = \frac{z}{(z-1)(z+1)^2}$ at its poles. **3**

(c)Show that **5**

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{3}.$$

6. State and prove argument principle theorem.

7. State and prove Rouche's theorem. Also determine the number of zeros of the polynomial $z^{10} - 6z^7 + 3z^3 + 1$ in $|z| < 1$.

8. State and prove Weierstrass's factorization theorem.

9. Derive Jensen's formula.

10. State and prove Hadamard's three circle theorem and hence prove that $\log \mathrm{M}(r)$ is a convex function of $\log(r)$

11. State and prove mean value theorem for an harmonic function.

12. Define analytic function. Show that an analytic function with constant modulus is constant.

13. State and prove Cauchy's inequality theorem.

14. State and prove Cauchy's theorem for a circular disc.

15. Expand the following function in Laurent's series valid for the region $1 < |z| < 3$.

$$f(z) = \frac{1}{(z+1)(z+3)}$$

16. Find the radius of convergence of the following functions.

(i) $\sum \left(\frac{2+in}{2^n}\right) z^n$

(ii) $\sum \left(\frac{2^{-n}}{1+in^2}\right) z^n$

(iii) $\sum \frac{(n!)^2}{(2n)!} z^n$

17. State and prove Taylor's theorem and expand $f(z) = \cos(z)$ about $z = \pi/4$.

18. Define the following terms and give one example for each.

(i) Isolated singularity.

(ii) Essential singularity.

19. Show that a function which has no singularity in the finite part of the plane or at infinity is constant.

20. Discuss the nature of singularity of the following functions.

(i) $f(z) = \sin z - \cos z$ at $z = \infty$

(ii) $f(z) = (z-3)\, sin \frac{1}{z+2}$ at $z = -2$

(iii) $f(z) = \frac{cot(\pi/z)}{(z-a)^2}$ at $z = \infty$.

21. (a) Evaluate the integral

$$\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta.$$

22. State and prove Weierstrass's factorization theorem.

23. State and prove open mapping theorem.

24. State and prove Schwarz's lemma.

25. Find the residue of the function at its poles

$$f(z) = \frac{1}{(z^2+1)^3}$$

26. Evaluate

$$\int_{\gamma} \frac{e^{2z}}{(z-1)^2(z+2)}\, dz$$

whereγ is a closed curve defined by $|z| = 4$ using the Cauchy's residue theorem.

27. State and prove maximum modulus theorem.

28. Derive Poisson-Jensen's formula.

29. State and prove Phragmen – Lindelof theorem.