**K.L.E Society's**
**S. Nijalingappa College**
**II BLOCK RAJAJINAGAR, BENGALURU -10**

## PG Department of Mathematics
## QUESTION BANK

# Differential Geometry

**1.** Given$V = y^2U_1 - xU_3, f - xy, g = z^3$, find
(i) $V[f]$and $V[g]$

(ii) $V[f^2 + g^2]$

(i) $V[fg]$

(ii) $V[V(f)]$

2. Define $p$ −differential forms on $\mathbb{R}^3$.
Evaluate the l-form $\phi = x^2dx - y^2dz$ on

(i) $V = xU_1 + yU_2 + z\mathrm{U}_3$

(ii) $W = xy(U_1 - U_2) + yz(U_1 - U_3)$

(iii) $\frac{1}{x}V + \frac{1}{y}W$.

3. Define

(i) Exterior derivative of a l-form on $\mathbb{R}^3$.

(ii) A mapping from $\mathbb{R}^m$to$\mathbb{R}^n$

4. Briefly explain the effect of the mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ on $\mathbb{R}^3$,where $F(u, v) = (u^2 - v^2, 2uv)$and $u, v$are coordinate functions of $\mathbb{R}^2$.

**5.** Briefly explain the effect of the mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ on $\mathbb{R}^3$,where $F(u, v) = (u^2 - v^2, 2uv)$and $u, v$are coordinate functions of $\mathbb{R}^2$.

If $F(u, v) = (u^2 - v^2, 2uv)$, find all the points $p$ for which

6. Write Frenet equations for unit speed curves and derive the same
7. Consider the unit speed curve
$\beta(s) = \left(\frac{(1+s)^{3/2}}{3}, \frac{(1-s)^{3/2}}{3}, \frac{s}{\sqrt{2}}\right)$ Find the Frenet apparatus for the same.

**8.** Prove the following

*(i)* $\nabla_{av_p+bw_p}X = a\nabla_{v_p}X + b\nabla_{w_p}X$

*(ii)* $\nabla_{v_p}(aX+bY) = a\nabla_{v_p}X + b\nabla_{w_p}Y$

*(iii)* $\nabla_{v_p}(X.Y) = \nabla_{v_p}X.Y(p) + X(p).\nabla_{v_p}Y$

$\nabla_{v_p}(fX) = v_p(f)X(p) + f(p)\nabla_{v_p}X$

9. Define

(i) An isometry of $\mathbb{R}^3$

(ii) Translation mapping

10. Further prove that every translation is an isometry and the composite mapping G°F of two isometries 'G' and 'F' is again an isometry.

11. Define a co-ordinate patch and a proper patch in $\mathbb{R}^3$.Prove that the surface $M$ abtained by translating a regular curve along some line $'l'$ is a surface in $\mathbb{R}^3$

12. Explain the parametrization of Torus in $\mathbb{R}^3$. Further use the parametrization to show that curve $(z-2)^2 + y^2 = 1$ revolved around z-axis is a Torus

13. Establish that the stereographic projection of the punctured sphere Σ onto the plane is a mapping

14. Let $F: M \to N$ be a mapping of surfaces and let $\xi$ and $\eta$ be $p$ −forms on $N$ then prove that

(i) $F^*(\xi + \eta) = F^*(\xi) + F^*(\eta)$

(ii) $F^*(\xi \Lambda \eta) = F^*(\xi) \Lambda F^*(\eta)$

$F^*(d\xi) = d(F^*(\xi)$

**15.** Define shape operator and find the shape operators of

Plane (ii) Cylinder

16. Define Gaussian curvature and mean curvature of a surface in$\mathbb{R}^3$. Under what conditions do we call a surface flat or minimal? Explain.

17. Show that

(i) $S(v) \times S(w) = K(p)(v \times w)$

18. $\big(S(v) \times S(w)\big) + \big(v \times S(w)\big) = 2\big(H(p)\big)(v \times w)$where the terms carry their usual meaning.

19. Compute the Gaussian and mean curvature of a saddle surface

20. Show that helicoid is a minimal surface.

21. Define (i) Tangent vector (ii) Vector field in $\mathbb{R}^3$.

22. Let $V_1 = U_1 - xU_{3,}V_2 = U_2,\ V_3 = xU_1 + U_{3,}$ then express $xU_1 + yU_2 + zU_3$ as a linear combination of $V_1, V_2, V_3$

23. Define directional derivative of a real valued function on $\mathbb{R}^3$.

Let $v_p$ and $w_p$ be two tangent vectors to $\mathbb{R}^3$ at $p$, $f$ and $g$ be real valued differentiable functions on $\mathbb{R}^3$ and $a, b \in \mathbb{R}$, then, prove that

(j) $[av_p + bw_p]f = a\, v_p[f] + bw_p[f]$

(jj) $v_p[af + bg] = av_p[f] + bv_p[g]$

$v_p(f.g) = v_p[f].g(p) + f(p).v_p[g]$

**24.** Define reparametrization of a curve in $\mathbb{R}^3$. Further explain the concept of instantaneous velocity of curve by visualizing it as a moving point.

**25.** Explain how a straight line and a helix are curves in $\mathbb{R}^3$?

**26.** If $\phi$ and ψ are l-forms than prove that $d(\phi\Lambda\psi) = d\phi\Lambda\psi = \phi\Lambda\psi$.

**27.** State Frenet formulae. Further explain the geometrical and physical meanings of all the Frenet apparatus, $T, N, B, \kappa, \tau$.

**28.** Find the Frenet apparatus for the unit speed helix $\beta(s) = \left(acos\frac{s}{c}, asin\frac{s}{c}, b\frac{s}{c}\right)$ where $c = \sqrt{a^2 + b^2}$.

**29.** If $C: \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation then show that $C$is an isometry on $\mathbb{R}^3$.

**30.** Compute connection forms and connection equations of cylindrical frame fields given by$E_1 = cos\theta U_1 + sin\theta U_2$ , $E_2 = sin\theta U_1 + cos\theta U_2$, $E_3 = U_3$.

**31.** Prove that a unit sphere Σ is a surface in $\mathbb{R}^3$.

**32.** Write a short note on patch computations including details of $u$ and $v$ parameter curves and partial velocities.Further establish the parametrization of a surface obtained by revolving a curve around an axis.

**33.** Define rulings of a surface. Show that the Saddle surface $M: z = xy$ is a doubly ruled surface.

**34.** Let $M$ be a right circular cone parametrized by $X(u, v) = (vcosu, vsinu, v)$

**35.** Let $\alpha$ be a curve such that $\alpha(t) = (t\sqrt{2}, e^t)$ on $M$ the find $\alpha'(t)$ in terms of $X_u$ and $X_v$. Also show that at each point of $\alpha, \alpha'$ bisects the angle between $X_u$ and $X_v$.

**36.** If $p$is an umbilic point on a surface $M$ than porve that the shape operator $S$ at $p$is just a scalar multiplication by the normal curvature

$$K = K_1 = K_2.$$

**37.** If $p$ is a non umbilic point then prove that there exists exactly two principal directions which are orthogonal and further prove that if $e_1$and $e_2$ are principal vendors in these directions them $\mathrm{S}(e_1) = ke_1$and$\mathrm{S}(e_2) = k_2e_2$

**38.** Define normal curvature of a surface in $\mathbb{R}^3$. Brief on how does determining the normal section for a surface explainthe shape of surface in $\mathbb{R}^3$

**39.** Compute the Gaussian and mean curvature,of a helicoid.

**40.** Define geodesics of a surface in $\mathbb{R}^3$. Obtain the geodesics for a plane, sphere and cylinder in $\mathbb{R}^3$.