K.L.E Society's
S. Nijalingappa College
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PG Department of Mathematics
QUESTION BANK

Functional Analysis

1. Define a normed linear space. Prove that addition and scalar multiplication in the normed linear space V are continuous

2. Define Banach space. Show that a real line $\mathbb{R}$ is Banach space.

3. Let N and N' be normed linear spaces with the same scalars and a linear transformation $T: N \to N'$. Then show that the following are equivalent to each other: **7**

(i) T is continuous on N.

(ii) T is continuous at the origin.

(i) T is bounded.

4. Let N be a non-zero normed linear space and $M = \{x/x \in N \text{ and } \|x\| = 1\}$. Then prove that N is a Banach space if and only if M is complete.

5. State and prove Hanh Banach theorem for real normed linear space.

6. Prove that N^* separates vectors in N, where N^* is the conjugate space of N.

7. State open mapping theorem and deduce closed graph theorem.

8. If P is a projection on a Banach space B and M and N are its range and null spaces respectively, then show that M and N are closed linear subspaces of B such that $B = M \oplus N$.

9. State and prove that parallelogram law and polarization identity holds in a Hilbert space.

10. Let M be a closed linear subspace of a Hilbert space H. Let $x \in H$ with $x \notin M$ and $d = d(x, M)$. Then show that there exists a unique $y_o \in M$ such that $\|x - y_0\| = d$.

11. If M is a closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^{\perp}$, where $M^{\perp}$ is an orthogonal complement of M.

12. Define complete orthonormal set. Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H. Then show that the following conditions are equivalent to one another.

i. $\{e_i\}$ is complete.

ii. $x \perp \{e_i\} \Rightarrow x = 0$.

iii. If x is an arbitrary vector in H, then $x = \sum_i \langle x, e_i \rangle e_i$.

iv. If x is an arbitrary vector in H, then $\|x\|^2 = \sum_i |\langle x, e_i \rangle|^2$.

13. Define self adjoint operator on a Hilbert space H. If A_1 and A_2 are self adjoint operators on H, then show that A_1A_2 is self adjoint if and only if $A_1A_2 = A_2A_1$.

14. If T is an arbitrary operator on a Hilbert space, then prove the following:

i. $T = 0$ if and only if $\langle Tx, y \rangle = 0$, for all $x, y \in H$

ii. $T = 0$ if and only if $\langle Tx, x \rangle = 0$, for all $x, y \in H$.

15. If N_1 and N_2 are normal operators on a Hilbert space H with the property that either of them commutes with the adjoint of other, then prove that $N_1 + N_2$ and N_1N_2 are normal.

16. If P is a projection on a Hilbert space H with range M and null space N, then show that M is orthogonal to N if and only if P is self adjoint and in this case $N = M^{\perp}$, where $M^{\perp}$ is the orthogonal complement of M

17. State and prove

18. Cauchy – Schwartz inequality (ii) Minkowski inequality

19. Show that the collection $B(X)$ of all bounded functions on a set X is a complete normed linear space.

20. Show that if $T: N \to N'$ is a linear transformation between normed linear spaces N and N', then prove that the following are equivalent.

(i) T is continuous

(ii) T is continuous at the origin

21. . Let M be a closed linear subspace of a normed linear space N. Show that $\frac{N}{M}$ is a normed linear space with the norm defined by $\| x + M \| = inf \{\| x + m \| \,/\, m \in M\}$.Also if N is a Banach space then prove that $\frac{N}{M}$ is also a Banach space.

22. If $S: N \to N'$ is a continuous linear transformation and M a null space. Show that S induces a natural linear transformation S' of $\frac{N}{M}$ to N' such that $\| S \| = \| S' \|$.

23. Let M be a linear subspace of a normed linear space N. If $x_0 \in N$ with $x_0 \notin M$ and

$M_0 = M + [x_0]$, then prove that f can be extended to a functional f_0 on M_0 such that$||f_0|| = ||f||$.

24. If N is a normed linear space and $x_0 \in N, x_0 \neq 0$, then prove that there is a $f_0 \in N^*$with $\| f_0 \| = 1$. Hence deduce that N^* separates vectors in N.
25. Prove that there is a natural imbedding of N into N^{**} by the isometricisomorphism $\varphi: N \to N^{**}$ defined by $\varphi(x) = F_x$, for all x in N where$F_x(f) = f(x) \forall f$ in N^*
26. State and prove closed graph theorem
27. Show that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm
28. If B is a complex Banach space whose norm obeys the parallelogram law and if an inner product is defined on B by $4 < x, y >= \| x + y \|^2 - \| x - y \|^2 + \mathrm{i} \| x + \mathrm{i}y \|^2 - \mathrm{i} \| x - \mathrm{i}y \|^2$for all $x, y \in B$, thenshow that B is a Hilbert space.
29. State and prove Gram Schmidt orthogonalizationtheorem
30. State and prove Bessel's inequality.
31. If A_1 and A_2 are self-adjoint operators on H, then prove that their product A_1A_2 is self-adjoint if and only if $A_1A_2 = A_2A_1$.
32. If 0 and I are zero and identity operators on H, then prove that $0^* = 0$ and $I^* = I$.

    Hence show that if T is non – singular operator on H then $T^*$is also non – singular
33. Let T be a normal operator on a Hilbert space H and f a polynomial with complex coefficients. Then prove that the operator $f(T)$ is normal.
34. If T is an operator on a Hilbert space H, then prove that the following conditions are all equivalent to one another:

    i) $T^*T = I$

    ii) $< Tx, Ty > = < x, y >$ for all $x, y \in H$

    $\| Tx \| = \| x \|$, for all $x \in H$.
35. If $P_1, P_2, \dots, P_n$are projections on closed linear subspaces $M_1, M_2, \dots, M_n$ of a Hilbert space H, then prove that $P_1 + P_2 + \dots + P_n$ is a projection if and only if $P_i's$ are pairwise orthogonal.
36. State and prove spectral theorem.