K.L.E Society's
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PG Department of Mathematics
QUESTION BANK
Linear Algebra:

1. Define algebra of a linear transformation. If A is an algebra over the field F then A is isomorphic to subalgebra of A(V) for some vector space V over field F.
2. Define minimal polynomial. If V is finite dimensional vector space over F then T∈A(V) is invertible iff the constant term of the minimal polynomial for T is non-zero.

3. Define rank of T. If V is finite dimensional vector space over F then for S,T ∈A(V)
   r(ST) ≤ r(T)
   r(TS) ≤ r(S)
   r(ST) = r(TS) = r(T) for S is singular.
4. Define characteristic root .If λ ∈ F is Characteristic root of T ∈ A(V) then for any q(x) ∈ F[x] , q(λ) is characteristic root of q(T).
5. State and prove Cauchy- Schwarz inequality.
6. State and prove Gram - Schmidt orthogonolization Process.
7. Define algebra. If A is an algebra with element over F, then prove that A is isomorphic to a subalgebra $A(V)$ of some vector space over F.
8. Let T and S be any two linear transformations of $A(V)$. If S is regular then show that T and STS^{-1} have the same minimal polynomials.
9. If V is a finite dimensional vector space over F then show that $T \in A(V)$ is invertible if and only if the constant term for T is non-zero.
10. If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct characteristic roots of $T \in A(V)$ and if $v_1, v_2, \dots, v_k$ are the corresponding characteristic vectors then prove that $v_1, v_2, \dots, v_k$ are linearly independent vectors.
11. Define a regular linear transformation. If V is a finite dimensional vector space over F then prove that $T \in A(V)$ is regular if and only if T maps V onto V.
12. Let V be the set of all polynomials in x of degree $n-1$. Let D be a linear operator defined by $(\beta_1 + \beta_2 x + \cdots + \beta_n x^{n-1})D = \beta_2 + 2\beta_3 x + \cdots + (n-1)\beta_n x^{n-2}$ where $D = \frac{d}{dx}$. Find the matrix corresponding to basis $\{1, x, x^2, \dots, x^{n-1}\}$ and $\{1, 1+x, 1+x^2, \dots, 1+x^{n-1}\}$.
13. If V is an n-dimensional vector space over F and if $T \in A(V)$ has a matrix $m_1(T)$ in the basis $\{v_1, v_2, \dots, v_n\}$ and a matrix $m_2(T)$ in the basis $\{w_1, w_2, \dots, w_n\}$ of V, then prove that there is a matrix $c \in F_n$ such that $m_2(T) = c\, m_1(T) c^{-1}$.
14. If W is a subspace of V which is invariant under T, then T induces a linear transformation $\bar{T}$ on $\bar{V}$ defined by $\bar{T}(v+w) = T(v) + W$. If T satisfies $q(x) \in F[x]$ then prove that $\bar{T}$ also satisfies $q(x)$. Further if $p_1(x)$ and $p(x)$ are minimal polynomials for $\bar{T}$ and T over F respectively then show that $p_1(x) \mid p(x)$. \

15. Define a nilpotent transformation. Prove that two nilpotent transformations are similar if and only if they have the same invariants.
16. Let $T \in A(V)$ contains all its distinct characteristic roots $\lambda_1, \lambda_2, \dots \lambda_k \in F$ then show that there is a basis of V in which the $m(T) = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & J_k \end{bmatrix}$ where each $J_i = \begin{bmatrix} B_{i_1} & 0 & \dots & 0 \\ 0 & B_{i_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & B_{i_{r_1}} \end{bmatrix}$ are basic Jordan blocks belonging to λ_i.
17. Define double dual of a vector space. Prove that double dual of V is isomorphic to V.
18. Verify if the following matrix is diagonalizable or not

$$\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}.$$

19. Define an inner product space. State and prove Cauchy Schwartz inequality.
20. Using Gram Schmidt process, construct an orthonormal basis from $\{(1,2,-2,4), (1,1,1,6), (5,2,2,5)\}$.
21. Classify the following quadratic forms:

    i) $2x_1^2 - 3x_1 x_3 + 8x_2 x_3 - 10x_2 x_1 - 7x_3^2$

    ii) $2x_1^2 + 5x_2^2 + 9x_3^2 - 2x_1 x_2 + 6x_3 x_1 + 6x_3 x_2$

    iii) $-3x_1^2 \, 8x_1 x_2 - 6x_2^2$.

22. Decompose the given matrix $\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$ using Singular Value Decomposition.
23. Prove that a bilinear form for a finite dimensional vector space is skew symmetric if and only if its matrix in some ordered basis is skew symmetric.
24. State and prove Sylvester's law of inertia.
25. Define minimal polynomial. If V is finite dimensional vector space over F then T∈A(V) is invertible iff the constant term of the minimal polynomial for T is non-zero.
26. Define rank of T. If V is finite dimensional vector space over F then for S,T ∈A(V)
    i. r(ST) ≤ r(T)
    ii. r(TS) ≤ r(S)
    iii. r(ST) = r(TS) = r(T) for S is singular.
27. Define characteristic root .If $\lambda \in$ F is Characteristic root of T ∈ A(V) then for any q(x) ∈ F[x] , q(λ) is characteristic root of q(T).
28. State and prove Cauchy- Schwarz inequality.
29. State and prove Gram - Schmidt orthogonolization Process.
30. Define minimal polynomial. Let T and S be any two linear transformations of $A(V)$. If S is regular then show that T and STS^{-1} have the same minimal polynomials.
31. Prove that $\lambda \in F$ is a characteristic root of $T \in A(V)$ if and only if for $v \in V, vT = \lambda v$.
32. If V is a finite dimensional vector space over F then show that $T \in A(V)$ is singular if and only if there exists a non-zero $v \in V$ such that $vT = 0$.

33. Define regular linear transformation. If V is finite dimensional vector space over F then for $S,T \in A(V)$ prove the following

    i) $r(ST) \leq r(T)$

    ii) $r(ST) = r(TS) = r(T)$ for $S \in A(V)$ is regular.

34. If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct characteristic roots of $T \in A(V)$ and if $v_1, v_2, \dots, v_k$ are the corresponding characteristic vectors then prove that $v_1, v_2, \dots, v_k$ are linearly independent vectors.
35. Let $B_1 = \{v_1, v_2, \dots, v_n\}$ and $B_2 = \{w_1, w_2, \dots, w_n\}$ be two bases for a vector space of dimension n. Suppose P is the change of basis from B_1 to B_2 and Q is the change of basis from B_2 *to* B_1 then prove that $P^{-1} = Q$. Illustrate this for the bases $B_1 = \{(1,2),(3,5)\}$ and $B_2 = \{(1,-1),(1,-2)\}$.
36. Let V be n-dimensional vector space with basis $B = \{v_1, v_2, \dots, v_n\}$. Then prove that there exists a uniquely determined basis $B^* = \{f_1, f_2, \dots, f_n\}$ such that $f_i(v_j) = \delta_{ij}$. Consequently, show that the dual space of an n-dimensional vector space is n- dimensional.
37. Prove that a matrix A of order $n \times n$ over a field F is digonalizable if and only if A has n linearly independent characteristic vectors in $V^n(F)$.
38. If $T \in A(V)$ has all its characteristic roots in F then prove that there exists a basis of V in which the matrix of T is triangluar.
39. Verify if the following matrix is diagonalizable or not

$$\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 1 \\ -2 & 4 & 1 \end{bmatrix}.$$

40. If $T \in A(V)$ is nilpotent transformation then prove that $\alpha_0 + \alpha_1 T + \alpha_2 T^2 + \dots + \alpha_m T^m$ is invertible if $\alpha_0 \neq 0$ for $\alpha_i \in F$.
41. Define a Jordan Canonical form with an example.
42. Using Gram Schmidt process, construct an orthonormal basis from $\{(2,0,1),(3,-1,5),(0,4,2)\}$.
43. Let $B = \{u_1, u_2, \dots, u_m\}$ be a finite orthonormal set in an inner product space V. If $v \in V$ show that $\sum_{i=1}^{m} |< v, u_i >|^2 \leq \|v^2\|$. Furthermore prove that equality holds if and only if v is in the subspace spanned by $\{u_1, u_2, \dots, u_m\}$.
44. For the matrix $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$, find the minimum value of the quadratic form subject to the constraint $x^T x = 1$ and the unit vector at which this value is attained.
45. Decompose the matrix $\begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & 2 \end{pmatrix}$ using Singular Value Decomposition.
46. Define a bilinear form. Compute the matrix of the given bilinear form $f((x_1, x_2, x_3), (y_1, y_2)) = -7x_1y_1 - 10\,x_1y_2 - 2x_2y_1 - 3x_2y_2 + 12\,x_3\,y_1 + 17x_3\,y_2$ with respect to the bases $U = \{(1,0,0),(1,1,0),(1,1,1)\}$ and $= \{(1,-1),(2,-1)\}$.
47. Define symmetric and skew symmetric bilinear forms.
48. Define rank and signature of a quadratic form and hence evaluate the rank and signature of the quadratic form $2x_1^2 + 6x_2x_1 + x_2^2$.