K.L.E Society's
S. Nijalingappa College
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PG Department of Mathematics
QUESTION BANK

Measure & Integration

1. (i) Define outer measure of a set prove that $m^*(A+x) = m^*(A)$ for every set A and every real number $'x'$.
   (ii) Show that for all real number x.
2. Prove that the outer measure of an interval is its length.
3. (a) If E_1 and E_2 are disjoint measurable subsets of set of all real numbers R then prove that $m^*[A \cap (E_1 \cup E_2)] = m^*(A \cap E_1) + m^*(A \cap E_2)$.
   (b) Define an algebra of sets prove that $\bigcap_{iz1}^{n} E_i$ belongs to algebra of sets.
   (c) Define Lebesgue measure
   (d) If $\{E_i\}$ is disjoint sequence of measureable sets, then prove that Lebesue measure is countable additive.
4. Let $\{E_i\}_{iz1}^{\infty}$ be an infinite decreasing sequence of measurable subsets of set of all real numbers. Let $m(E_i) < \infty$ for at least one i belonging to set of all natural number IN. Then prove that $m(\bigcap_{iz1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$
5. Define
   (i) Measureable function.
   (ii) Characteristic function.
6. Let E be a measurable set then show that characteristic function χ_A is measureable if and only if A is measurable.
7. If $f: E \to \mathbb{R}^*$ is a measureable function and C is any real number then show that $C \pm f$ and Cf are measureable and hence prove that $-f$ is measureable where $\mathbb{R}^*$ is set of all positive real numbers.
8. State and prove Egoroff's theroem.
9. Let $\{E_i\}_{i_2}^{n}$, be a finite disjoint collection of measurable subsets of a set of finite measure E for $1 \leq i \leq n$, let a_i be a real number. If $\emptyset = \sum_{i=1}^{n} a_i \chi_{E_i}$ on E then show that $\int_E \emptyset = \sum_{i=1}^{n} a_i m(E_i)$
10. If f is a non-negative measureable function on E, then show that for any

$$\lambda > 0, m\{x \in E | f(x) \geq \lambda\} \leq \frac{1}{\lambda} \int_E f$$

11. The f and g are integrable over E, then prove that for any two real numbers $\propto$ and β, the function $\propto f + \beta g$ is integrable over E and

$$\int_E \propto f + \beta g = \propto \int_E f + \beta \int_E g \text{ and if } f \leq g \text{ then } \int_E f \leq \int_E g$$

12. (a) State and prove Fatou's Lema.

(b) Evaluate the Lebesgue integral of the function $f: [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{\sqrt[3]{x}} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

(c) If f is integrable over E and $\{E_n\}_{n=1}^{\infty}$ is a disjoint countable collection of measurable subsets of E whose union is E then. Prove that

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f$$

13. Establish Vitali covering Lemma

14. Define a function of bounded variation on an interval $[a, b]$. Prove that a function f is of bounded Variation on $[a, b]$ if and only if it is the difference of two increasing functions on $[a, b]$.

15. Defined an absolutely continuous function on an interval $[a, b]$ show that a Lipschitz function on $[a, b]$ is absolutely continuous.

16. Let E be a measurable set. $1 \leq p < \infty$ and q the conjugate of p. If $f \in L^p(E)$ and $g \in L^q(E)$ show that f_g is integrable over E and

$$\int_E |fg| \leq \|f\|_p \|g\|_q$$

If $f \neq 0$ and $f^* = \|f\|_p^{1-p}$ of $|f|^{p-1}$ then show that

$$\int_E ff^* = \|f\|_p \text{ and } \|f^*\|_q = 1 \text{ where } = \begin{cases} 1 \text{ if } f(x) \geq 0 \\ -1 \text{ if } f(x) < 0 \end{cases}$$

17. If q is the conjugate of p where $1 < p < \infty$ and $g \in L^q(E)$ show that the functional $Fg: L^p(E) \to \mathbb{R}$ defined by $Fg(f) = \int_E fg$ for all $f \in L^p(E)$ is bounded line as functional on $L^p(E)$ and $\|Fg\| = \|g\|_q$.

18. Define outer measure of set A. Show that the outer measure of an interval is its length.

19. If S_1 and S_2 are measurable sets, then prove that $S_1 \cup S_2$ is measurable.

20. Show that every Borel set in $\mathbb{R}$ is measurable.

21. Prove that Lebesgue measure is countably additive.

22. Let $\{E_i\}$ be an infinite decreasing sequence of measurable subsets of $\mathbb{R}$ and $m(E_1)$ is finite then show that $m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n)$.

23. Define measurable function. Let $f: E \to \mathbb{R}$be a function where E is a measurable set then prove that f is measurable if and only if for any open set G in $\mathbb{R}$, $f^{-1}(G)$ is a measurable.

24. If $\{f_n\}$ converges in measure to f then prove that the limit function f is unique almost everywhere.

25. Let E be a measurable set with $m(E) < \infty$ and $\{f_n\}$ be a sequence of measurable functions defined on E. Let f be a measurable function such that $f_n \to f$ on E. Then show that given $\epsilon > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $m(A) < \delta$ and an integer N such that $|f_n(x) - f(x)| < \epsilon \forall x \in E - A$ and $\forall n \geq N$.

26. If f and g are bounded measurable functions defined on a set E of finite measure then show the following results are true:

i) $\int af = a\int f$

ii) $\int_E f + g = \int_E f + \int_E g$

27. State and prove Lebesgue Monotone convergence theorem.

Let $\{E_i\}$ be a sequence of disjoint measurable sets with $\bigcup_{i=1}^{\infty} E_i = E$and f is non negative measurable function then prove that $\int_E f = \sum_{i=1}^{\infty} \int_{E_i} f\, dx$

28. Let f be a non negative integrable function over a measurable set E then prove that given $\epsilon > 0 \; \exists \; \delta > 0$ such that for every set $A \subseteq E$ with $m(A) < \delta$ we have $\int_A f < \epsilon$.

29. State and prove Generalized Lebesgue convergence theorem.

30. Let f be a function of bounded variation on $[a, b]$ then show that $T_a^b = P_a^b + N_a^b$and $f(b) - f(a) = P_a^b - N_a^b$ where T_a^b, P_a^b and N_a^b have their usual meaning.

31. Let f be bounded, measurable function on$[a, b]$ and $F(x) = \int_a^x f(t)\,dt + F(a)$ then prove that $F'(x) = f(x)$ almost everywhere $\forall x \in [a, b]$.

32. Define outer measure of a set. Prove that outer measure is,

i. Translation invariant.

ii. Countable sub-additive

33. Define measurable sets. Prove that if E_1, E_2 are measurable sets then $E_1 \cup E_2$ is also measurable.

34. Show that the interval (a, ∞) is measurable.

35. Let $\{E_i\}_{i=1}^{\infty}$ be an infinite decreasing sequence of measurable subsets of $\mathbb{R}$, where

E_i is of finite measure for atleast one i in $\mathbb{N}$. Prove that $m(\cap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$.

36. Prove the following properties of measurable functions.

(i) If f is a measurable function on a measurable set E and $E_1 \subset E$ then f is measurable on E_1.

(ii) If f is a measurable function on each of the sets in a countable collection $\{E_i\}$ of disjoint measurable sets then f is measurable on $\bigcup_{i=1}^{\infty} E_i$.

(iii) If f and g are measurable functions on a common domain then the set $A(f, g) = \{x \in A; f(x) < g(x)\}$ is measurable

37. Prove that a continuous function defined on measurable set is measurable.

38. Let f, g be functions from E to $\mathbb{R}^*$ the set of extended real numbers, be measurable functions then show that $f \pm g: E \to \mathbb{R}^*$ are also measurable

39. Prove that every step function is measurable.

40. Let f and g be two bounded measurable functions defined on a set of finite measure E. The prove $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$, where α & β are real numbers. Also show that if $f \leq g$ on E, then $\int_E f \leq \int_E g$.

41. State and prove Lebesgue dominated convergence theorem.

42. Prove that a function F is an indefinite integral if and only if it is absolutely continuous.