K.L.E Society's
S. Nijalingappa College
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QUESTION BANK

ODE

1. Let $\{\phi_j(x);\ j = 1, 2, \dots, n\}$ be 'n' solutions of $L_n y = 0$ in the interval I satisfying the initial conditions $\phi_j^{(i-1)}(x_0) = b_{ij}$, where $(1 \le i, j \le n)$, $x_0 \in I$ and b_{ij} are constants. Prove that a necessary and sufficient condition for the set $\{\phi_j(x); j = 1, 2, \dots, n\}$ for a fundamental set for $L_n\, y = 0$ is $|b_{ij}| \neq 0$.
2. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of $y'' + a(x)\, y = 0$, then show that $z(x) = y_1(x)y_2(x)$ satisfies the differential equation $z''' + 4a(x)z' + 2a'(x)\, z = 0$.
3. State and prove Abel – Liouville theorem
4. Find the adjoint differential equation of $x^2y'' - 2xy' + 3y = 0$, then show that it is a self-adjoint differential equation.
5. Solve $y''' - 6y'' + 11y' - 6y = e^x$ by using the variation of parameters method.
6. State and prove Sturm's comparison theorem on the zeros of a self – adjoint differential equation
7. Find the eigenvalues and eigenfunctions of $\frac{d}{dx}\{xy'\} + \frac{\lambda}{x} y = 0$ in $[1, e^{2\pi}]$; $y'(1) = 0 = y'(e^{2\pi})$.
8. Find the Green's function of $y'' + \lambda y = b(x);\ y(0) = 0 = y(\pi)$ in $[0, \pi]$.
9. Define ordinary and singular points of $y'' + P(x)y' + Q(x)y = 0$. Find the power series solution of $y'' + (x - 3)y' + y = 0$ about $x = 2$.
10. Find the polynomial solution of Laguerre differential equation
11. State and prove orthogonal property for Hermite polynomials.
12. Find the general solution of Gauss hypergeometric equation
13. Prove the following.

    i) $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0, (n \ge 1)$

    ii) $(1 - x^2)\, T_n'(x) = -nxT_n(x) + nT_{n-1}(x), (n \ge 1)$

14. Find the fundamental matrix of $\frac{dx}{dt} = 4x - y;\ \ \frac{dy}{dt} = x + 2y.$

15. If $\phi(t)$ is a fundamental matrix of $\frac{dX(t)}{dt} = A\,X(t)$on $[a, b]$, then prove that

$$\phi_0(t) = \phi(t) \int_{t0}^{t} \phi^{-1}(u)F(u)du,$$ where $t \in [a, b]$ is a particular solution of

$$\frac{dX(t)}{dt} = A\,X(t) + F(t).$$

16. Determine the nature and stability of the critical point (0, 0) of the following systems.

i) $\frac{dx}{dt} = 2x - 7y;\ \frac{dy}{dt} = 3x - 8y.$

ii) $\frac{dx}{dt} = -x - 2y;\ \frac{dy}{dt} = 4x - 5y.$

17. Using the Liapunov method, determine the stability of the critical point (0, 0) of

$$\frac{dx}{dt} = -2xy;\ \frac{dy}{dt} = x^2 - y^3.$$

18. Define fundamental set. Let $\{\phi_j(x);\ j = 1, 2, \ldots, n\}$ be a fundamental set of $L_n y = 0$, then show that $\{\psi_j(x);\ j = 1, 2, \ldots, n\}$ also forms a fundamental set of $L_n y = 0$ if and only if there exists a non-singular constant matrix C such that $[\psi_1\ \psi_2\ \ldots\ \psi_n]^T = C\,[\phi_1\ \phi_2\ \ldots\ \phi_n]^T$. Also deduce that $W\{\psi_j(x);\ j = 1, 2, \ldots, n\} = |C|\,W\{\phi_j(x);\ j = 1, 2, \ldots, n\}$.

19. Find the Wronskian of $y''' - y'' - y' + y = 0.$

20. Verify Liouville's formula for $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0.$

21. Solve the equation $x^2 y'' + 7xy' + 8y = 0$ by finding the solution of its adjoint equation.

22. State and prove Sturm's separation theorem on zeros of a self-adjoint differential equation. Illustrate it with an example.

23. Show that $y'' + \frac{k}{x^2} y = 0$, where k is a constant and $x > 0$ is oscillatory if $k > \frac{1}{4}$ and is non-oscillatory if $k \le \frac{1}{4}$.

24. Define Sturm – Liouville problem. Find the eige Define self-adjoint eigenvalue problem. For such a problem prove that,

i) the eigenvalues are real,

ii) the eigenvalues corresponding to two distinct eigenvalues are orthogonal nvalues and eigenfunctions of the Sturm – Liouville problem $y'' + \lambda y = 0;\ \ y(0) = 0 = y'(\pi)$ in $(0, \pi)$.

25. Find the power series solution in powers of $(x-1)$ of the initial value problem $xy'' + y' + 2y = 0$; $y(1) = 1, y'(1) = 2$.

26. Define regular singular point of the differential equation

$y'' + p(x)y' + Q(x)y = 0$. Solve $2xy'' + 3y' - y = 0$ by using the Frobenious method.

27. Prove the following:

i) $H_n'(x) = 2n\, H_{n-1}(x)$,

ii) $H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x)$. **5**

(b) State and prove generating function for Laguerre polynomial. **5**

(c) Prove that **4**

$$\int_{-1}^{+1} \frac{T_m(x)\, T_n(x)}{\sqrt{1-x^2}}\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi/2 & \text{if } m = n \neq 0 \\ \pi & \text{if } m = n. \end{cases}$$

28. Show that $X(t) = Ce^{At}$ is a general solution of the equation $\frac{dX(t)}{dt} = A\,X(t)$.

Hence deduce the solution of the equation when $A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$. **8**

(b)Find the critical point and determine its nature and stability of

$$\frac{dx}{dt} = 2x + 7y; \quad \frac{dy}{dt} = 3x + 8y.$$

29. Explain the four types of critical points of the system

$$\frac{dx}{dt} = ax + by; \quad \frac{dy}{dt} = cx + dy, \text{where } ad - bc \neq 0.$$

30. Using the Liapunov method, determine the stability of the critical point (0, 0) of

$$\frac{dx}{dt} = -x^5 - y^3; \quad \frac{dy}{dt} = 3x^3 - y^3.$$