K.L.E Society's
S. Nijalingappa College
II BLOCK RAJAJINAGAR, BENGALURU -10

PG Department of Mathematics
QUESTION BANK

Real Analysis

1. Establish the definition and existence of Riemann - Stieltje's integral and hence prove the Darboux condition of integrability.

2. Find the upper and lower Riemann – Stieltje's sums of the function $f(x) = 2x + 1$ with respect to $\alpha(x) = x$ corresponding to the division of the interval [0 , 1] into 5 subintervals of equal length.

3. If'f'is a Riemann – Stieltje'sintegrable function with respect to a function 'α' on $[a, b]$ and c belongs to (a, b) then prove that f is a Riemann – Stieltje's integrable function on $[a, c]$and $[c, b]$.

4. If f_1, f_2 belong to $\Re\,(\alpha)$ on $[a, b]$ then prove that $f_1 + f_2$ belongs to $\Re\,(\alpha)$ and also show that $\int_a^b f_1 + f_2\; d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$where the terms carry their usual meaning.

5. If f belongs to $\Re(\alpha)$ for the partition P and s_i, t_i be arbitrary points in $[x_{i-1}, x_i]$ then prove that,

$i)$ $\sum_{i=1}^{n} |f(s_i) - f(t_i)\Delta\alpha_i| < \varepsilon$

$ii)$ $\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \varepsilon$

6. Define a mesh of an interval and $S(P, f, \alpha)$. Further prove that the existence of the limit of $S(P, f, \alpha)$, as the mesh tends to 0, is a sufficient condition for f to belong to $\Re(\alpha)$. Also show that the $\lim\limits_{\mu(P)\to 0} S(P, f, \alpha) = \int_a^b f d\alpha$.

7. Define uniform convergence of a sequence of real valued functions. Further check the uniform convergence of the following sequence.

$i)$ $\{f_n\} = \frac{x}{n}$, where x belongs to [0, 1].

$ii)$ $\{f_n\} = \left\{\frac{nx}{1+n^3x^2}\right\}$, where x belongs to [0, 1].

8. Let α be a monotonically increasing functions on $[a, b]$. Suppose f_n is Riemann – Stieltje'sintegrable function with respect to α on $[a, b]$ for each n and converges

to f uniformly on $[a, b]$, then prove that f is Riemann – Stieltje'sintegrable function with respect to α on $[a, b]$. Further show that

$$\int_a^b f d\,\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha.$$

9. Show that an infinite series $\sum f_n(x)$ converges uniformly on E if and only if for all $\varepsilon > 0$ there exists N such that $n > N$ implies $\left|\sum_{K=n+1}^{n+p} f_k(x)\right| < \varepsilon$ for all $p = 1, 2, 3, \ldots$ and x in E.

10. State and prove the Weierstrass M test to check the convergence of an infinite series of real valued function.

11. Test the uniform convergence of the series $\sum_{n=0}^{\infty}(1-x)x^n$, where x belongs to $[0, 1]$.

12. Suppose K is subsest of Y which is subset of X then show that K is compact relative to X if and only if K is compact relative to Y.

13. Prove that $\bigcap_{n=1}^{\infty} I_n$ is non-empty for *n=1,2,3,....* and when $I_{n+1} \subset I_n$, if $\{I_n\}$ is a sequence of:

i. Intervals in $\mathbb{R}$.

ii. K– cells, where K is a positive integer.

14. Define a differentiable function of several variables. If 'f' maps an open set E in $\mathbb{R}^n$ into $\mathbb{R}^m$ and f is differentiable at a point x in E, prove that the partial derivatives $D_j f_i(x)$ exists and $f'(x)\, e_j = \sum_{i=1}^{m}(D_j f_i)\ (x) u_i$,where $1 \le j \le n$ and $u_i's$, $e_j's$ are standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

15. Define a contraction mapping. If $\phi: X \longrightarrow X$ is a contraction on a complete metric space X then prove that ϕ has a unique fixed point.

16. State and prove inverse function theorem.

17. Show that the condition $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ for every $\varepsilon > 0$, is the necessary and sufficient condition for a function 'f' to be Riemann – Stieltje's integrable function with respect to α on $[a, b]$.

18. Show that $f(x) = x^2$ belongs to $\Re(x^3)$ on $[0,\ 1]$, where the terms carry their usual meaning.

19. If'f'is a Riemann – Stieltje'sintegrable function with respect to the function α_1 and α_2 on $[a,b]$ then prove that f is Riemann – Stieltje's integrable function with respect to $\alpha_1 + \alpha_2$on$[a,b]$.

20. Prove that $-f$ belongs to $\Re\,(\alpha)$if f belongs to $\Re\,(\alpha)$on $[a,b]$, where the terms carry their usual meaning.

21. If f belongs to $\Re([a,b])$and α is a monotonically increasing function on$[a,b]$such that α' belongs to $\Re([a,b])$ then prove that$f$ belongs to $\Re\,(\alpha)$ and further establish that $\int_a^b f\,d\alpha = \int_a^b f\alpha' d\alpha$, where the terms above carry their usual meaning.

22. If f is a continuous function on $[a,b]$ and ϕ is a continuous and strictly monotonic function on $[\alpha,\beta]$, where $a = \phi\,(\alpha)$and $b = \phi(\beta)$. Then prove that

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(y))d\phi(y)$$

23. State and prove Cauchy's criterion for uniform convergence of a sequence of real valued functions.

24. Let $\{f_n\}$ be a sequence of functions differentiable on $[a,b]$ such that $\{f_n(x_o)\}$ converges for some point $x_o$in $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$ then prove that $\{f_n\}$ converges uniformly on $[a,b]$ to a function f and also establish that $f'(x) = \lim_{n\to\infty} f_n'(x)$where $a \le x \le b$.

25. State and prove Wierstrass M test.

26. Use theWierstrass M test to check the convergence of the following series

$i)$ $\sum_{n=0}^{\infty}(1-x)x^n$, for all $x\ in[0,1]$;

$ii)$ $\sum_{n=0}^{\infty}\frac{\cos nx}{n^2}$ where x belongs to $\mathbb{R}$ and $\left|\frac{\cos nx}{n^2}\right| \le \frac{1}{n^2}$.

27. Suppose Y is a subset of X. Show that A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset $G$of X.

28. Let E be a subset of $\mathbb{R}^k$ then prove that the following statements are equivalent.

i) $E$is closed and bounded.

ii) E is compact.

iii) Every infinite subset of E has a limit point in E.

29. Define Hessain matrix. State and prove Rank theorem.

30. State and prove the Implicit function theorem.